arXiv:1007.2970v1 [math.AP] 18 Jul 2010

EVENTUAL REGULARITY OF THE SOLUTIONS TO THE SUPERCRITICAL DISSIPATIVE QUASI-GEOSTROPHIC EQUATION

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ABSTRACT. Recently in [4], Silvestre proved that certain weak solutions of the slightly supercritical surface quasi-geostrophic equation eventually become smooth. To prove this, he employed a De Giorgi type argument originated in the work of Caffarelli and Vasseur, [3]. In [5], Kiselev and Nazarov proved a variation of the result of Caffarelli and Vasseur by introducing a class of test functions. Motivated by the results of Silvestre, we will modify the class of test functions from [5] and use this modified class to show that a solution to the supercritical SQG that is smooth up to a certain time must remain smooth forever.

1. INTRODUCTION

The setting of this paper will be the d-dimensional torus, $\mathbb{T}^d$. We may equivalently think of the problem in the setting of $\mathbb{R}^d$ with periodic initial data. Throughout the paper we will consider only real valued functions. We consider the Cauchy problem for the dissipative equation

$$\begin{cases} \theta_t = (u \cdot \nabla)\theta - (-\Delta)^{\alpha/2}\theta \\ \theta(x,0) = \theta_0(x) \end{cases}, \tag{1}$$

where $u = R\theta$, R is a certain divergence free operator, and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian. In the case of the surface quasi-geostrophic equation (SQG for brevity), $d = 2$ and $u = (-R_2\theta, R_1\theta)$, where the R_js are the standard Riesz transforms. These operators are defined on a suitably smooth class of functions by multiplication on the Fourier side. For $n \in \mathbb{Z}^d$, if

$$\widehat{\theta}(n) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \theta(x)e^{-in\cdot x}\,dx$$

is the n^{th} Fourier coefficient of θ, then for $n \neq 0$

$$\widehat{(R_j\theta)}(n) = i\frac{n_j}{|n|}\widehat{\theta}(n) \quad \text{and} \quad [(-\Delta)^{\alpha/2}\theta]^\frown(n) = |n|^\alpha\widehat{\theta}(n),$$

and $\widehat{(R_j\theta)}(0) = [(-\Delta)^{\alpha/2}\theta]^\frown(0) = 0$.

The parameter α ranges between 0 and 2. The case when $\alpha \in (1, 2]$ is referred to as the subcritical case. In the subcritical case, the global well-posedness has been established in the case of smooth initial data (See [1] and the references therein). The critical case, $\alpha = 1$, has been the source of much study in recent years. In [1], Constantin, Cordoba, and Wu proved that if the L^∞ norm on the initial data is small enough, then there is a global regular solution. Later, Kiselev, Nazarov, and Volberg introduced the modulus of continuity method in [6]. This method was used to prove the global well posedness of the critical SQG for smooth periodic initial data by finding a priori bounds on $\|\nabla\theta\|_\infty$. In the supercritical case, $\alpha < 1$, many open questions remain.

In [3], Caffarelli and Vasseur used De Giorgi iteration to show that a uniform bound in BMO of the velocity term in the drift diffusion equation implies that certain weak solutions are locally Hölder continuous. In [5], Kiselev and Nazarov showed that, in the case of the critical surface quasi-geostrophic equation, a uniform bound in BMO on a smooth velocity leads to a certain degree of Hölder continuity. In this way they were able to give yet another proof of the existence of global smooth solutions to the critical surface quasi-geostrophic equation. Their approach relies on passing the evolution onto a special class of functions which is "dual" to the class of Hölder continuous functions. The reason for studying the Hölder continuity of solutions to the SQG can be seen from results of Constantin and Wu [2]. They showed that if you have a uniform bound on the $C^{1-\alpha+\delta}$ $(\delta > 0)$ norm of a certain weak solution to the SQG on a time interval, then in fact you have a smooth classical solution on that interval. Recently in [4], Silvestre proved that if the dissipative power is slightly smaller than 1/2, namely the power of the Laplacian is $\frac{1-\epsilon}{2}$ for small ϵ, then certain weak solutions become Hölder continuous after a certain time. The proof employs De Giorgi-type estimates to show that on a parabolic cylinder the oscillation of a certain continuation of the solution is not more than a fraction of the oscillation of the continuation of the solution on a twice larger parabolic cylinder under the assumption that the $L^{2d/\alpha}$ norm of the velocity is uniformly bounded.

Before stating the main result, we define

$$\|f\|_{k,p} = \|\nabla^k f\|_p,$$

which for mean zero functions can be shown to be equivalent to the standard Sobolev norm by the Poincaré inequality. Motivated by the smooth class constructed in the work of Nazarov and Kiselev and the work of Silvestre, we will prove the following theorem:

Theorem 1. *Suppose that R is a divergence free vector-valued operator that, for every $k \geq 0$ and every $1 < p < \infty$, satisifes $\|Rf - Rg\|_{k,p} \leq C(k,p)\|f - g\|_{k,p}$ for some constants $C(k,p)$, and, for every $\epsilon > 0$ satisifes $\|\nabla(Rf)\|_\infty \leq C(\epsilon)\|\nabla f\|_{C^\epsilon}$ for some constant $C(\epsilon)$. There is a time $T = T(\alpha, \|\theta_0\|_\infty)$ such that if $\theta \in C^\infty(\mathbb{T}^d \times [0,T])$ is a solution to the Cauchy problem*

$$\begin{cases} \theta_t = (R\theta \cdot \nabla)\theta - (-\Delta)^{\alpha/2}\theta \\ \theta(x,0) = \theta_0(x) \end{cases},$$

then θ extends to a solution in $C^\infty(\mathbb{T}^d \times [0,\infty))$.

A consequence of this theorem is

Theorem 2 (Eventual Regularization for the Supercritical SQG)**.** *There is a time $T = T(\alpha, \|\theta_0\|_\infty)$ such that if $\theta \in C^\infty(\mathbb{T}^2 \times [0,T])$ is a solution to*

$$\begin{cases} \theta_t = (R^\perp\theta \cdot \nabla)\theta - (-\Delta)^{\alpha/2}\theta \\ \theta(x,0) = \theta_0(x) \end{cases},$$

then θ extends to a solution in $C^\infty(\mathbb{T}^2 \times [0,\infty))$.

Classical results about Riesz transforms imply $R^\perp$ satisfies the conditions in Theorem 1 (See [7]). Both Theorems tell us that for any value of α in the supercritical range, if we have a solution that is smooth up to a certain time, then it remains smooth forever.

2. Dualizing the Problem

We now define a variant of the class introduced in [5]. Let $A > 1$ be a parameter to be fixed later.

Definition 3. *We will say that a smooth function ψ defined on $\mathbb{T}^d$ is in $\mathcal{U}(r)$ if*

$$\int_{\mathbb{T}^d} |\psi(x)|^p \, dx \leq A r^{-(p-1)d} \tag{2}$$

and

$$\sup \Big\{ \Big| \int_{\mathbb{T}^d} f(x)\psi(x)\, dx \Big| : f \in C^\infty \cap Lip(1) \Big\} \leq r \tag{3}$$

In this definition we have used the notation $Lip(M)$ to denote the class of all functions f such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{T}^d$. Since all constant functions are Lipschitz, if (3) holds, then the function ψ must have mean zero. Also notice that if a function ψ satisfies $\|\psi\|_1 \leq 1$ and $\|\psi\|_\infty \leq A^{1/(p-1)} r^{-d}$ (as were the conditions in [5]), then interpolation shows that ψ satisfies (2). If φ is supported in B_r (the ball of radius r centered at the origin in $\mathbb{T}^d$), has mean zero and $\|\varphi\|_p \leq r^{-d/q}$, then $\varphi \in \mathcal{U}(r)$. In what follows we will write $f \in B\,\mathcal{U}(r)$ if $f/B \in \mathcal{U}(r)$.

Recall that a function g is Hölder continuous with exponent $\beta \in (0,1)$ if $|g(x) - g(y)| \leq C|x-y|^\beta$ for some constant C and all $x, y \in \mathbb{T}^d$. We will denote the class of Hölder continuous functions with exponent β on $\mathbb{T}^d$ by $C^\beta(\mathbb{T}^d)$. Paley-Littlewood projections can be used to characterize $C^\beta(\mathbb{T}^d)$ as follows: we let ω be a smooth compactly supported function on $\mathbb{R}^d$ that is identically 1 when $|x| \leq 1$, radially decreasing, and vanishing for $|x| \geq 2$. Define $\varphi(x) = \omega(x) - \omega(2x)$ and $\varphi_j(x) = \varphi(x/2^j)$. For an integrable function f on $\mathbb{T}^d$, we define for any non-negative integer j

$$\Delta_j f(x) = \sum_{n \in \mathbb{Z}^d} \varphi_j(n) \widehat{f}(n) e^{in\cdot x}.$$

The operators Δ_j are essentially smooth projections on the frequency scale 2^j. Recall that a bounded function g on $\mathbb{T}^d$ is $C^\beta(\mathbb{T}^d)$ if and only if for every $j \geq 0$,

$$\|\Delta_j g\|_\infty \leq M 2^{-\beta j}, \tag{4}$$

with some $M > 0$. See [8] for a proof. Write $\langle f, g\rangle$ to denote $\int_{\mathbb{T}^d} f(x)g(x)\, dx$. If we have control over

$$|\langle g, \mathcal{U}(r)\rangle| := \sup \big\{ |\langle g, \psi\rangle| : \psi \in \mathcal{U}(r) \big\},$$

then we get control over the Hölder C^β seminorm of g. More precisely, we have the following

Lemma 4. *Suppose that a function g on $\mathbb{T}^d$ has the property that*

$$\Big| \int_{\mathbb{T}^d} g(x)\psi(x)\, dx \Big| \leq r^\beta,$$

for all $\psi \in \mathcal{U}(r)$ and $0 < r \leq 1$. Then $g \in C^\beta(\mathbb{T}^d)$ and

$$\|g\|_{C^\beta} := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\beta} \leq C(\beta).$$

Proof. Let φ_j be the function defined above and let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Since φ is smooth, compactly supported, and vanishes at the origin, $\mathcal{F}^{-1}\varphi$ is a Schwartz function with mean zero. We can therefore find a constant C such that $\int_{\mathbb{R}^d} |\mathcal{F}^{-1}\varphi(x)|\,dx \leq C$, $\int_{\mathbb{R}^d} |x|\cdot|\mathcal{F}^{-1}\varphi(x)|\,dx \leq C$, and $|\mathcal{F}^{-1}\varphi(y)| \leq C(1+|y|^2)^{-d}$ for all $y \in \mathbb{R}^d$. Scaling these inequalities gives for all $j \geq 0$,

(i) $\int_{\mathbb{R}^d} |\mathcal{F}^{-1}\varphi_j(x)|\,dx \leq C$
(ii) $\int_{\mathbb{R}^d} |x|\cdot|\mathcal{F}^{-1}\varphi_j(x)|\,dx \leq C2^{-j}$
(iii) $|\mathcal{F}^{-1}\varphi_j(y)| \leq C2^{jd}(1+|2^j y|^2)^{-d}$ for all $y \in \mathbb{R}^d$

Now define for $x \in \mathbb{T}^d$,

$$\Phi_j(x) = c\sum_{n\in\mathbb{Z}^d}(\mathcal{F}^{-1}\varphi_j)(x+2\pi n),$$

for some constant c to be chosen later. We claim that if we choose c sufficiently small independently of j, then $\Phi_j \in \mathcal{U}(2^{-j})$. Inequality (iii) implies that $\|\Phi_j\|_\infty \leq cC'2^{jd}$ and inequality (i) implies $\|\Phi_j\|_1 \leq cC'$ for some constant $C' > C$. Interpolation shows the norm condition, (2), is satisfied provided c is small. Inequality (ii) and mean zero property of $\mathcal{F}^{-1}\varphi_j$ show that

$$\left|\int_{\mathbb{R}^d} f(x)\mathcal{F}^{-1}\varphi_j(x)\,dx\right| \leq C2^{-j},$$

for any smooth 2π-periodic function f in $Lip(1)$ on $\mathbb{R}^d$. This implies that the smooth condition, (3), is satisfied, provided c is small enough. Since Δ_j is a convolution with Φ_j on the space side, we have for any $y \in \mathbb{T}^d$,

$$\int_{\mathbb{T}^d} g(x)\Phi_j(x-y)\,dx = c\Delta_j g(y).$$

Since the class $\mathcal{U}(r)$ is invariant under translations $\Phi_j(\cdot - y) \in \mathcal{U}(2^{-j})$ for any $y \in \mathbb{T}^d$, which implies the left hand side of the above equality is not more than $2^{-j\beta}$ by assumption. Therefore, we have $\|\Delta_j g\|_\infty \leq c^{-1}2^{-j\beta}$ for all $j \geq 0$. From (4) we conclude that $g \in C^\beta(\mathbb{T}^d)$. □

If we wish to prove that the solution to the SQG at time t is in C^β, we must estimate $|\langle\theta(\cdot,t),\mathcal{U}(r)\rangle|$. We do so by determining how the class $\mathcal{U}(r)$ evolves under the backward equation. Let $u = R\theta$ and suppose that ψ is a solution to the equation

$$\psi_s = (u\cdot\nabla)\psi + (-\Delta)^{\alpha/2}\psi \tag{5}$$

Later we will impose some "future condition" that at some moment of time, the solution to (5) is in the class $\mathcal{U}(r)$. Consider the pairing function defined for $\tau > 0$,

$$P(\tau) = \langle\theta(\cdot,\tau),\psi(\cdot,\tau)\rangle = \int_{\mathbb{T}^d}\theta(x,\tau)\psi(x,\tau)\,dx$$

Equations (1) and (5) together with the fact that u is a divergence free vector field imply that P has zero derivative and is therefore constant:

$$\int_{\mathbb{T}^d}\theta(x,t-s)\psi(x,t-s)\,dx = \int_{\mathbb{T}^d}\theta(x,t)\psi(x,t)\,dx, \tag{6}$$

for any pair of times $t \geq s \geq 0$. Our next objective will be to find functions F and G such that if ψ is a solution to (5) and $\psi(\cdot,t) \in \mathcal{U}(r)$ for some fixed time t, then $\psi(\cdot,t-s) \in$

$F(s,r)\mathcal{U}(G(s,r))$ for s sufficiently small. This is what is meant by dualizing the problem: (6) allows us to move the dynamics from a solution to (1) onto a solution of (5). Now we can determine the Hölder regularity of the solution to (1) by determining how (5) alters the class $\mathcal{U}(r)$.

3. Evolution of the Class $\mathcal{U}(r)$

The heart of the proof of Theorem 1 is the following

Lemma 5. *(Class Evolution) Given $\alpha, \beta \in (0,1)$ and $p > 1$ such that $\alpha + \beta - d/q > 1$ where q is the conjugate exponent to p, there are parameters $\delta, r_0 > 0$ with the following property: if $0 < r \leq r_0$, $s \leq r^\alpha$, and $|\langle \theta(\cdot,\tau), \mathcal{U}(R)\rangle| \leq R^\beta$ for all $R \geq re^\delta$ and $\tau \in [t-s,t]$, then every solution ψ to (5) with $\psi(\cdot,t) \in \mathcal{U}(r)$ satisfies*

$$\psi(\cdot, t-s) \in \exp(-\delta s r^{-\alpha}) \mathcal{U}\big(r \exp(\delta\beta^{-1} s r^{-\alpha})\big). \tag{7}$$

Proof. Let $\widetilde{\chi}$ be a smooth radially decreasing function on $\mathbb{R}^d$ supported in $\{x : |x| \leq 1\}$ and has mean 1. Then define $\widetilde{\chi}_r(x) = r^{-d}\widetilde{\chi}(x/r)$, so that $\widetilde{\chi}_r$ has mean 1 for every $r > 0$ and $\|\nabla \widetilde{\chi}_r\|_\infty \leq C r^{-d-1}$. For $r \in (0,1]$, the function $\widetilde{\chi}_r$ is supported in $(-\pi,\pi)^d$. Using the identification of $\mathbb{T}^d$ with $[-\pi,\pi]^d$, we identify $\widetilde{\chi}_r$ with a function χ_r on $\mathbb{T}^d$.

Suppose now that $f = f(x,\tau)$ is a solution to the smooth forward evolution:

$$\begin{cases} f_\tau = (u_r \cdot \nabla) f - (-\Delta)^{\alpha/2} f \\ f(x, t-s) = f_0(x) \end{cases}, \tag{8}$$

where $u_r = R(\theta * \chi_r)$ and R is the divergence free operator in the statement of Theorem 1. Consider the pairing function Π defined in the interval $[t-s,t]$ by

$$\Pi(\tau) = \langle f(\cdot,\tau), \psi(\cdot,\tau)\rangle = \int_{\mathbb{T}^d} f(x,\tau)\psi(x,\tau)\,dx,$$

where ψ is a solution to (5). Differentiating Π and using that u_r and u are divergence free, we see that

$$\Pi'(\tau) = \int_{\mathbb{T}^d} [(u_r(x,\tau) - u(x,\tau)) \cdot \nabla f(x,\tau)]\psi(x,\tau)\,dx.$$

Integrating the above expression we have

$$\int_{\mathbb{T}^d} \psi(x,t-s) f_0(x)\,dx = \int_{\mathbb{T}^d} \psi(x,t) f(x,t)\,dx + \int_{t-s}^t \int_{\mathbb{T}^d} [(u-u_r)(x,\tau)\cdot \nabla f(x,\tau)]\psi(x,\tau)\,dx\,d\tau. \tag{9}$$

Call the absolute value of the first integral on the right hand side of the above equation the *smooth part*, which we denote by I, and the absolute value of the second integral the *rough part*, which we denote by II. Then we have

$$\left| \int_{\mathbb{T}^d} \psi(x,t-s) f_0(x)\,dx \right| \leq I + II.$$

In what follows we will estimate the quantities I and II.

4. Modulus of Continuity Redux: The Smooth Part

Using a modulus of continuity argument inspired by [6], we will determine the size of the smooth part.

Lemma 6. *(Lipschitz Evolution) Let v be a smooth divergence free vector field such that $v(x,\tau) \in Lip(M)$ for all $\tau \in [t-s,t]$. Suppose that f is a solution to the system*

$$\begin{cases} f_\tau = (v\cdot\nabla)f - (-\Delta)^{\alpha/2}f \\ f(x,t-s) = f_0(x) \end{cases},$$

where f_0 is a smooth function in $Lip(1)$. Then $f(x,t-k) \in Lip(\exp(M(s-k)))$, for all $k \in [0,s]$.

Proof. Fix $\epsilon > 0$ and consider

$$\kappa = \sup\left\{k\in[0,s] : \exists x,y\in\mathbb{T}^d \text{ such that } |f(x,t-k)-f(y,t-k)| \geq e^{M(s-k)}(|x-y|+\epsilon)\right\}.$$

The global regularity theory for an equation of the form (8) with smooth velocity implies that f is smooth for all times. In what follows we will omit the absolute value signs around the quanitity $f(x,\tau)-f(y,\tau)$ as we may always make it is non-negative by exchanging x and y if necessary. Suppose $\kappa \geq 0$. First, we notice that $\kappa \neq s$. Indeed, if it were, then there would be sequences of points $x_n, y_n \in \mathbb{T}^d$ and $k_n \to s$ such that

$$f(x_n,t-k_n)-f(y_n,t-k_n) \geq \exp(M(s-k_n))(|x_n-y_n|+\epsilon).$$

The compactness of $\mathbb{T}^{2d}$ implies there are points x and y such that

$$f(x,t-s)-f(y,t-s) \geq |x-y|+\epsilon,$$

which contradicts the assumption on f_0. It follows that for $\kappa < k < s$ and all $x,y\in\mathbb{T}^d$ we have

$$f(x,t-k)-f(y,t-k) < \exp(M(s-k))(|x-y|+\epsilon).$$

Passing to the limit as k tends to κ in the previous inequality, the continuity of f implies that for all $x,y\in\mathbb{T}^d$,

$$f(x,t-\kappa)-f(y,t-\kappa) \leq \exp(M(s-\kappa))(|x-y|+\epsilon).$$

Using the same compactness argument as above, we see that there are points $x,y\in\mathbb{T}^d$ such that

$$f(x,t-\kappa)-f(y,t-\kappa) = \exp(M(s-\kappa))(|x-y|+\epsilon). \tag{10}$$

We now claim that there is a $k > \kappa$ such that at these points $x,y\in\mathbb{T}^d$ we have

$$f(x,t-k)-f(y,t-k) \geq \exp(M(s-k)(|x-y|+\epsilon). \tag{11}$$

To this end, we will now compute

$$\partial_k(f(x,t-k)-f(y,t-k))\Big|_{k=\kappa}. \tag{12}$$

The velocity term is the derivative of f in the direction of v; more precisely, the chain rule gives $(v\cdot\nabla f)(x) = \frac{d}{dh}f(x+hv(x))|_{h=0}$. At the breaking points x and y we have

$$f(x+hv(x),t-\kappa)-f(y+hv(y),t-\kappa) \leq \exp(M(s-\kappa))[(|x-y|+h|v(x)-v(y)|)+\epsilon].$$

Subtracting $\exp(M(s-\kappa))(|x-y|+\epsilon)$ from both sides, dividing by h, and passing to the limit gives

$$(v\cdot\nabla f)(x,t-\kappa)-(v\cdot\nabla f)(y,t-\kappa)\leq \exp(M(s-\kappa))M|x-y|.$$

The next contribution to (12) comes from the dissipative term. Consider the pure dissipative equation

$$\begin{cases} g_\tau = -(-\Delta)^{\alpha/2}g \\ g(\cdot,t-\kappa)=f(\cdot,t-\kappa) \end{cases}.$$

The solution to this equation is $g(z,\tau)=f(\cdot,t-\kappa)*\Phi(z,\tau)$, where $\widehat{\Phi}(\xi,\tau)=\exp(-|\xi|^\alpha\tau)$. The estimates on f at time $t-\kappa$ imply

$$g(z_1,t-\kappa)-g(z_2,t-\kappa)\leq \exp(M(s-\kappa))(|z_1-z_2|+\epsilon),$$

for all $z_1,z_2\in\mathbb{T}^d$. Since the solutions to the purely dissipative equation perserve the modulus of continuity, $g(x,\tau)-g(y,\tau)\leq\exp(M(s-\kappa))(|x-y|+\epsilon)$ for all $\tau\geq t-\kappa$. The contribution of the dissipative part to (12) is exactly the same as $\partial_\tau(g(x,\tau)-g(y,\tau))|_{\tau=t-\kappa}$. Since $g(x,t-\kappa)-g(y,t-\kappa)=\exp(M(s-\kappa))(|x-y|+\epsilon)$ and $g(x,\tau)-g(y,\tau)\leq\exp(M(s-\kappa))(|x-y|+\epsilon)$ for all $\tau\geq t-\kappa$, the function $g(x,\tau)-g(y,\tau)$ has a local maximum at $\tau=t-\kappa$. It follows that $\partial_\tau(g(x,\tau)-g(y,\tau))|_{\tau=t-\kappa}\leq 0$ and

$$[-(-\Delta)^{\alpha/2}f](x,t-\kappa)-[-(-\Delta)^{\alpha/2}f](y,t-\kappa)\leq 0.$$

Combining the estimates for the velocity and the dissipation we see that

$$\partial_k(f(x,t-k)-f(y,t-k))\Big|_{k=\kappa}\geq -\exp(M(s-\kappa))M|x-y| \tag{13}$$

The k derivative of the growth condition, $\exp(M(s-k))(|x-y|+\epsilon)$, at the point κ is $-M\exp(M(s-\kappa))(|x-y|+\epsilon)$, which is strictly smaller than the right hand side of (13). It follows that for k slightly larger than κ we have (11), which contradicts the choice of κ. From this we conclude

$$\left\{k\in[0,s]:\exists x,y\in\mathbb{T}^d \text{ such that } |f(x,t-k)-f(y,t-k)|\geq e^{M(s-k)}(|x-y|+\epsilon)\right\}$$

is empty for every $\epsilon>0$ and the lemma follows. □

We now wish to apply the previous lemma to the solution f of (8). Since $\theta_r=\theta*\chi_r$ is a smooth function, the velocity term $u_r=R(\theta_r)$ is a C^1 divergence free vector field. In order to find a uniform bound on the Lipschitz constant of u_r, we must first estimate the Hölder norm of $\nabla\theta_r$. To this end, we notice that the choice of χ_r implies that $\nabla\chi_r$ is a mean zero vector-valued function supported in the set B_r and has L^∞ norm at most Cr^{-d-1}. It follows that $r\nabla\chi_r\in C\mathcal{U}(2r)$ (meaning each component is in $C\mathcal{U}(2r)$). By the assumption of the Class Evolution Lemma, after the time $t-s$ the solution pairs well against $\mathcal{U}(R)$ for $R\geq re^\delta$; therefore provided $\delta<1/2$, we see that for any $\tau\in[t-s,t]$ we have $\|\nabla\theta_r(\cdot,\tau)\|_\infty=\|\theta(\cdot,\tau)*\nabla\chi_r\|_\infty\leq C|\langle\theta(\cdot,\tau),r^{-1}\mathcal{U}(2r)\rangle|\leq Cr^{\beta-1}$. Similarly, $\|\nabla(\theta(\cdot,\tau)*\nabla\chi_r)\|_\infty\leq Cr^{\beta-2}$. Let $\epsilon>0$. Interpolation implies that the C^ϵ norm of $\nabla\theta_r$ is no more than $Cr^{\beta-1-\epsilon}$. The norm assumption on R implies $\|\nabla R\theta_r(\cdot,\tau)\|_\infty\leq Cr^{\beta-1-\epsilon}$. Thus, $R\theta_r(\cdot,\tau)\in Lip(Cr^{\beta-1-\epsilon})$ for any $\tau\in[t-s,t]$. The Lipschitz Evolution Lemma implies $f(\cdot,t)\in Lip(\exp(Csr^{\beta-1-\epsilon})$. Since $\psi(\cdot,t)\in\mathcal{U}(r)$ by assumption, we have the following estimate for the smooth part:

$$I\leq\left|\int_{\mathbb{T}^d}\psi(x,t)f(x,t)\,dx\right|\leq r\exp(Csr^{\beta-1-\epsilon}). \tag{14}$$

5. Mean Zero Duality: The Rough Part

In this section, we will estimate the rough part of the evolution. Recall that the rough part was the expression

$$II = \Big| \int_{t-s}^{t} \int_{\mathbb{T}^d} \big((u - u_r)(x,\tau) \cdot \nabla f(x,\tau)\big)\psi(x,\tau)\, dx\, d\tau \Big|.$$

Trivially estimating II by Hölder's inequality yields

$$II \leq s \sup_{\tau \in [t-s,t]} \big(\|(u-u_r)(\cdot,\tau)\|_q \|\psi(\cdot,\tau)\|_p \|\nabla f(\cdot,\tau)\|_\infty \big). \tag{15}$$

The maximum principle implies $\|\psi(\cdot,\tau)\|_p \leq \|\psi(\cdot,t)\|_p \leq A^{1/p} r^{-d/q}$ for $\tau \in [t-s,t]$. The Lipschitz Evolution Lemma implies that $\|\nabla f(\cdot,\tau)\|_\infty$ is not more than $\exp(Csr^{\beta-1-\epsilon})$. Since R is Lipchitz in the L^q norm, $\|(u-u_r)(\cdot,\tau)\|_q \leq C_q \|(\theta-\theta_r)(\cdot,\tau)\|_q$, so it suffices to bound $\|(\theta-\theta_r)(\cdot,\tau)\|_q$.

Recall that χ_r was chosen to have mean 1, so for any constant c

$$\|(\theta-\theta_r)(\cdot,\tau)\chi_{B_r}\|_q \leq \|(\theta-c)(\cdot,\tau)\chi_{B_r}\|_q + \|(c-\theta)_r(\cdot,\tau)\chi_{B_r}\|_q \leq 2\|(\theta-c)(\cdot,\tau)\chi_{B_{3r}}\|_q. \tag{16}$$

We claim that for some choice of c the above expression is not more than $Cr^{\beta+d/q}$. We will prove this using the smoothness on larger scales along with the following

Lemma 7 (Mean Zero Duality)**.** *For any $\rho > 0$, there is a constant c such that*

$$\Big(\int_{B_\rho} |\theta - c|^q\, dx \Big)^{1/q} \leq \sup\{\rho^{d/q} |\langle \theta, \psi \rangle| : \psi \in \mathcal{U}(\rho)\}. \tag{17}$$

Proof. Choose c so that $\operatorname{sgn}(\theta-c)|\theta-c|^{q-1}$ has mean zero on B_ρ and define $\lambda = \rho^{-d/q}\|(\theta - c)\chi_{B_\rho}\|_q^{-q/p}$. With these choices $\psi = \lambda \operatorname{sgn}(\theta-c)|\theta-c|^{q-1}\chi_{B_\rho}$ is a mean zero function supported in B_ρ. A direct computation shows $\|\psi\|_p^p \leq \rho^{-(p-1)d} \leq A\rho^{-(p-1)d}$ since $A > 1$. As mentioned previously, the mean zero, support, and norm properties of ψ imply the Lipshitz pairing condition. We know choose a sequence of smooth functions $\psi_j \in \mathcal{U}(\rho)$ which converge to ψ in L^p norm. Since $\rho^{d/q}|\langle\theta,\psi_j\rangle| \to \rho^{d/q}|\langle\theta,\psi\rangle|$ and the latter expression is left hand side of (17), the Lemma follows. $\square$

Applying the lemma to the left hand side of (16) with $\rho = 3r$ for any $\tau \in [t-s,t]$ gives

$$\Big(\int_{B_r} |(\theta-\theta_r)(x,\tau)|^q\, dx \Big)^{1/q} \leq C(3r)^{d/q} |\langle \theta, \mathcal{U}(3r)\rangle| \leq Cr^{d/q+\beta}.$$

Since $\mathbb{T}^d$ can be covered by a constant multiple of r^{-d} balls of radius r, adding the q^{th} powers of the left hand sides of the above inequalities for all these balls yields

$$\sup_{\tau\in[t-s,t]} \|(\theta-\theta_r)(\cdot,\tau)\|_q = \sup_{\tau\in[t-s,t]} \Big(\int_{\mathbb{T}^d} |(\theta-\theta_r)(x,\tau)|^q\, dx \Big)^{1/q} \leq Cr^{-d/q} r^{d/q+\beta} \leq Cr^\beta. \tag{18}$$

The above estimates, (15), and (18) imply

$$II \leq C_q C A^{1/p} r^{\beta - d/q} s \exp(Csr^{\beta-1-\epsilon}) \tag{19}$$

Adding the contributions of the smooth part (14) and the rough part (19) and choosing $\epsilon < d/q$, we have for all $\tau \in [t-s, t]$ and for some $C_q' > 0$,

$$\sup_{f_0 \in Lip(1)} \Big| \int_{\mathbb{T}^d} \psi(x,\tau) f_0(x)\, dx \Big| \le r \exp\big(C_q' A^{1/p} s r^{\beta-1-d/q}\big). \tag{20}$$

In particular, we have

$$\sup_{f_0 \in Lip(1)} \Big| \int_{\mathbb{T}^d} \psi(x,t-s) f_0(x)\, dx \Big| \le r \exp\big(C_q' A^{1/p} s r^{\beta-1-d/q}\big). \tag{21}$$

It follows that for $r \le r_0$, (21) is stronger than what we need for (7) provided

$$C_q' A^{1/p} r_0^{\beta-d/q-1+\alpha} \le \delta(\beta^{-1} - 1). \tag{22}$$

6. The Decay of the L^p Norm

In this part, we show the way it decays of the L^p norm on scale r is stronger than what we need for (7). Computing the derivative of the p^{th} power of the L^p norm of $\psi(\cdot,\tau)$ gives

$$\frac{d}{d\tau} \int_{\mathbb{T}^d} |\psi(x,\tau)|^p\, dx = p \int_{\mathbb{T}^d} \Psi(x,\tau)(-\Delta)^{\alpha/2}\psi(x,\tau)\, dx, \tag{23}$$

where $\Psi(x,\tau) = |\psi(x,\tau)|^{p-2}\psi(x,\tau)$ (here we used the fact that the velocity was divergence free). We also have the well-known formula

$$(-\Delta)^{\alpha/2}\psi(x,\tau) = C_\alpha \sum_{n\in\mathbb{Z}^d} \text{p.v.} \int_{\mathbb{T}^d} \frac{\psi(x,\tau) - \psi(y,\tau)}{|x-y-n|^{\alpha+d}}\, dy. \tag{24}$$

See [10] for a proof of (24). If we plug (24) into (23) and symmetrize, we see that the derivative of the p^{th} power of the L^p norm is

$$\frac{p}{2} C_\alpha \sum_{n\in\mathbb{Z}^d} \lim_{l\to 0} \int_{D_l} \frac{(\Psi(x,\tau) - \Psi(y,\tau))(\psi(x,\tau) - \psi(y,\tau))}{|x-y-n|^{\alpha+d}}\, dy\, dx, \tag{25}$$

where $D_l = \{(x,y) \in \mathbb{T}^d \times \mathbb{T}^d : |x-y| \ge l\}$. Notice that the integrand in (25) is non-negative. If $r \le r_)$, $s \le r^\alpha$, and $\delta\beta^{-1} < \log 2$, (20) implies that for the smooth function $\eta(\cdot,\tau) = \Psi(\cdot,\tau) * \chi_r$

$$\int_{\mathbb{T}^d} \psi(x,\tau)\eta(x,\tau)\, dx < 2r\|\nabla\eta\|_\infty = 2r\|\Psi(\cdot,\tau) * \nabla\chi_r\|_\infty \le 2r\|\Psi(\cdot,\tau)\|_q \|\nabla\chi_r\|_p. \tag{26}$$

The choice of χ_r implies that $2r\|\nabla\chi_r\|_p \le Cr^{-d}r^{d/p} = Cr^{-d/q}$, for some constant $C = C(\chi)$. Notice that $\|\Psi(\cdot,\tau)\|_q = \|\psi(\cdot,\tau)\|_p^{p-1}$.

We may assume $\|\psi(\cdot,t-s)\|_p \ge A^{1/p}\frac{r^{-d/q}}{2}$ (provided $\delta(1 + d(\beta q)^{-1}) \le \log 2$), otherwise the evolution would already be satisfied. The maximum principle implies $\|\psi(\cdot,\tau)\|_p \le \|\psi(\cdot,t)\|_p$ for any $\tau \in [t-s, s]$, so substituting the above inequalities into (26) yields

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \psi(x,\tau)\Psi(y,\tau)\chi_r(x-y)\, dy\, dx \le Cr^{-d/q}\|\psi(\cdot,\tau)\|_p^{p-1} \le 2CA^{-1/p}\|\psi(\cdot,\tau)\|_p^p. \tag{27}$$

The same inequality holds if x and y are interchanged. Let $I(x,y,\tau)$ be the numerator of the integrand in (25), then $I(x,y,\tau) \ge 0$ by the above comments. Notice that $|x-y|^{-\alpha-d} \ge c'r^{-\alpha}\chi_r(x-y)$, for all $x, y \in \mathbb{T}^d$ with some constant c' depending only on χ. Therefore, the

kernel in (25) dominates $c' r^{-\alpha}\chi_r(x-y)$. Leaving only the central cell contribution ($n=0$) in (25) and scaling Λ by $C_\alpha c' r^{-\alpha}\frac{p}{2}$ gives

$$\frac{d}{d\tau}\int_{\mathbb{T}^d}|\psi(x,\tau)|^p\,dx \geq C_\alpha\frac{p}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}I(x,y,\tau)|x-y|^{-\alpha-d}\,dx\,dy \geq C_\alpha c'\frac{p}{2}r^{-\alpha}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}I(x,y,\tau)\chi_r(x-y)\,dx\,dy$$

$I(x,y,\tau)$ is a sum of four terms: two of the form $\Psi(x,\tau)\psi(x,\tau)=|\psi(x,\tau)|^p$ and two of the form $-\Psi(x,\tau)\psi(y,\tau)$. Since χ_r has mean 1, the former terms contribute $C_\alpha c' p r^{-\alpha}\|\psi(\cdot,\tau)\|_p^p$. The latter terms contribute no less than $-2CC_\alpha c' A^{-1/p} p r^{-\alpha}\|\psi(\cdot,\tau)\|_p^p$ by (27) . Therefore, we have the lower bound

$$\frac{d}{d\tau}\int_{\mathbb{T}^d}|\psi(x,\tau)|^p\,dx \geq C_\alpha c' p(1-2CA^{-1/p})r^{-\alpha}\|\psi(\cdot,\tau)\|_p^p. \tag{28}$$

Provided we choose p first, we can choose A large enough so that

$$1-2CA^{-1/p} > 1/2 \tag{29}$$

and integrate the inequality (28) to get

$$\int_{\mathbb{T}^d}|\psi(x,t-s)|^p\,dx \leq Ar^{-(p-1)d}\exp(-cpsr^{-\alpha}), \tag{30}$$

with $c=c(\chi,\alpha,p)=C_\alpha c'/2$. It follows that for $r\leq r_0$ and $s\leq r^\alpha$, (30) is stronger than what we need for (7) provided

$$\delta \leq \min\{\beta\log 2, (1+d(\beta q)^{-1})^{-1}\log 2, (1+d(\beta q)^{-1})^{-1}c\} \tag{31}$$

This proves the Class Evolution Lemma provided δ and r_0 are small and A is large. □

7. The Proof of Theorem 1

If we are given a solution θ to (1) with initial data whose mean is $\bar{\theta}_0\neq 0$, we define $\widetilde{\theta}(x,t)=\theta(x,t)-\bar{\theta}_0$ and an operator $\widetilde{R}(\varphi)=R(\varphi+\bar{\theta}_0)$. The modified operator $\widetilde{R}$ still satisfies the assumptions of Theorem 1. Furthermore, $\widetilde{\theta}$ is mean zero and θ solves (1) if and only if $\widetilde{\theta}$ solves the equation

$$\widetilde{\theta}_t = (\widetilde{R}\widetilde{\theta}\cdot\nabla)\widetilde{\theta} - (-\Delta)^{\alpha/2}\widetilde{\theta}.$$

If we can show there is a time T such that every $\widetilde{\theta}\in C^\infty(\mathbb{T}^d\times[0,T])$ can be extended to a function in $C^\infty(\mathbb{T}^d\times[0,\infty))$, then the same conclusion holds for θ. It follows that we may assume that θ_0 has mean zero.

Let $\alpha<1$, choose $\beta>1-\alpha$, and then choose $p>1$ so that $\beta+\alpha-d/q>1$ and $q=2^n$ for some positive integer n. Now we select the parameters from the Class Evolution Lemma. Choose A large enough so that (29) is true. Now, we choose δ small enough so that (31) is true. Finally, we choose r_0 sufficiently small so (22) is true.

Since the initial data θ_0 has mean zero, the maximum principle implies that the L^q norm of θ decays exponentially. More precisely, $\|\psi(\cdot,\tau)\|_q \leq C(\|\theta_0\|_\infty)\exp(-\tau/q)$. Indeed, the proof of Lemma 2.4 in [10] implies for θ_0 with mean zero,

$$\frac{d}{d\tau}\|\theta(\cdot,\tau)\|_q^q = -q\int_{\mathbb{T}^d}|\theta(x,\tau)|^{q-2}\theta(x,\tau)(-\Delta)^{\frac{\alpha}{2}}\theta(x,\tau)\,dx \leq -\int_{\mathbb{T}^d}|(-\Delta)^{\frac{\alpha}{4}}\theta^{\frac{q}{2}}(x,\tau)|^2\,dx.$$

Since $\widehat{\theta}(\cdot,\tau)(0)=0$, by passing to the Fourier side we see

$$\frac{d}{d\tau}\|\theta(\cdot,\tau)\|_q^q \le -\int_{\mathbb{T}^d} |\theta^{q/2}(x,\tau)|^2\,dx = -\|\theta(\cdot,\tau)\|_q^q.$$

This implies $\|\theta(\cdot,\tau)\|_q^q \le \|\theta_0\|_q^q \exp(-\tau) \le C\|\theta_0\|_\infty^q \exp(-\tau)$. It follows that there is a time T_0 (depending on $\|\theta_0\|_\infty$) such that$|\langle\theta(\cdot,\tau),\mathcal{U}(r)\rangle| \le r^\beta$ if $\tau \ge T_0$ and $r_0 \le r \le 1$. Define

$$T_k = T_0 + \beta r_0^\alpha \sum_{j=0}^{k-1} e^{-\delta\alpha j}.$$

We now claim that if $t \ge T_k$, then $|\langle\theta(\cdot,t),\mathcal{U}(r)\rangle| \le r^\beta$ for $r \ge r_0 e^{-\delta k}$. This is certainly true for $k=0$ by definition of T_0. Suppose that the claim is true for some k. Let $r \in [r_0 e^{-\delta(k+1)}, r_0 e^{-\delta k})$ and $s=\beta r^\alpha$. Suppose that $t \ge T_{k+1}$ and $\psi(\cdot,t) \in \mathcal{U}(r)$. Notice that $re^\delta \ge r_0 e^{-\delta k}$ and $t-s \ge T_k$. The Class Evolution Lemma and (6) imply

$$\Big|\int_{\mathbb{T}^d} \theta(x,t)\psi(x,t)\,dx\Big| = \Big|\int_{\mathbb{T}^d} \theta(x,t-s)\psi(x,t-s)\,dx\Big| \le e^{-\delta\beta}(re^\delta)^\beta \le r^\beta.$$

It follows that the claim is true for all $k \ge 0$. Passing to the limit, we see that θ pairs well against any $\mathcal{U}(r)$ after the moment

$$T = T(\alpha, \|\theta_0\|_\infty) = \lim_{k\to\infty} T_k = T_0 + \frac{\beta r_0^\alpha}{1-\exp(-\delta\alpha)}.$$

For any time $t \ge T$, we have

$$\Big|\int_{\mathbb{T}^d} \theta(x,t)\varphi(x)\,dx\Big| \le r^\beta, \tag{32}$$

for all $\varphi \in \mathcal{U}(r)$ and all $0 < r \le 1$. It follows that past the moment $T(\alpha,\|\theta_0\|_\infty)$, the solution is Hölder continuous with exponent β with Hölder norm uniformly bounded.

We have now shown that there is a time after which we have a uniform bound on the C^β norm on the solution for β as close to 1 as we wish. A generalization of the argument of Constantin, Cordoba, and Wu ([1]) shows that this is sufficient to conclude that the solution is smooth past this moment.

8. Concluding Remarks

The method presented above can be used to prove that a viscosity weak solution of (1) eventually becomes smooth and therefore eventually becomes a classical solution. The terminology and following definitions come from [10]. Given $\theta_0 \in C^\infty(\mathbb{T}^d)$, a viscosity solution to the (1) is a weak limit (in L^2) of solutions to

$$\begin{cases} \theta_t = (R\theta\cdot\nabla)\theta - (-\Delta)^{\alpha/2}\theta + \epsilon\Delta\theta \\ \theta(x,0) = \theta_0(x) \end{cases}, \tag{33}$$

as $\epsilon \to 0$. The pertubation by the Laplacian and smooth initial initial data guarantee a solution, θ^ϵ, to (33) is smooth for all times. The extra dissipative term doesn't affect the above estimates which allows us to conclude $\|\theta^\epsilon(\cdot,t)\|_{C^\delta} \le C$ with $\delta > 1-\alpha$ for all $t \ge T(\alpha,\|\theta_0\|_\infty)$ uniformly in ϵ. The results from [1] now give the desired regularity. In particular, the above argument gives another proof of the main result in [4].

Acknowledgment. The author is grateful to Fedor Nazarov for his guidance in the completion of this project.

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